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研究集会「組合せ論的表現論とその周辺」
複素単純 LIE 群の外部自己同型の重複度 1 の分岐則について,
MULTIPLICITY-FREE BRANCHING RULES
FOR OUTER AUTOMORPHISMS OF SIMPLE LIE ALGEBRAS

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We announce the paper [2].

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{g}'$ be a reductive Lie subalgebra of $\mathfrak{g}$. The restriction $\pi|_{\mathfrak{g}'}$ of a irreducible representation π of $\mathfrak{g}$ need not be irreducible.

The irreducible decompsiton of $\mathfrak{g}'$

$$\pi|_{\mathfrak{g}'} = \bigoplus_{\mu \text{is irreducible representation of } \mathfrak{g}'} c_\pi^\mu \mu$$

is called *branching rule.*

Problem 1. $\boxed{\textit{Say something about } c_\pi^\mu.}$

Exmaple 1. There are well-known branching rules.

(1) **classical rule.** Set $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $\mathfrak{g}' = \mathfrak{sl}_n$, then irreducible representations of $\mathfrak{g}$ are indexed by $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. In this case, we have the complete answer to Problem 1.

We have

$$\lambda|_{\mathfrak{g}'} = \bigoplus_{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \mu.$$

In particular, we have $c_\pi^\mu \leq 1$.

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(2) **highest weight theory.** Let $\mathfrak{g}'$ be a Cartan subalgebra of $\mathfrak{g}$. The answer of Problem 1 is the Cartan-Weyl's highest weight theory. The branching rule is decomposition of weight spaces. The description of c_π^μ is Kostant's formula.

(3) **tensor product** Let $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}'$ and $\mathfrak{g}'$ diagonal in $\mathfrak{g}$. The irreducible representation of $\mathfrak{g}$ is given by $\pi = \sigma \boxtimes \tau$ where σ and τ are irreducible representations of $\mathfrak{g}'$. The branching rule $\pi|_{\mathfrak{g}'} = \pi \otimes \sigma$ is the tensor product, which causes Littlewood-Richardson rule.

Remark 2. Koike–Terada [9] gave general formulas of $GL(n)$ to $SO(n)$ or $GL(2n)$ to $Sp(n)$ by using the universal characters.

Problem 2. $\boxed{\textit{In which cases do we have } c_\pi^\mu \leq 1?}$

This branching rule is called multiplicity-free.

Remark 3. It is difficalt to get the weight mutiplicity-free representations, though we have the Kostant's general formula. Similarly, we do not have the classification of mutiplicity-free branching rules, though we have the general formula Koike–Terada's algorigthm.

Exmaple 4. We have some answers to the emaples in Example 1.

(1) always
(2) few
(3) few (Multiplicity-free tensor products are called Clebsch–Gordan's rule, which are cllasified by Stembridge [13].

Remark 5. Kobayashi recently obtained an abstract theorem of multiplicity-free branching rules for both infinite and finite dimensional representations for a general symmetric pair (G, G') [6] [8].
Okada uses new combinatorial formulas on minors due to Ishikawa-Wakayama [5] to obtain explicit branching rules [11].

We want a new technique in getting many branching rules.

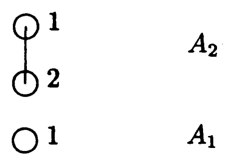
In the paper [2], we get the many examples of multiplicity-free branching rules, which we intoduce in this proceeding.

2. Setting.

Let $\mathfrak{g}$ be a complex simple Lie algebra, σ be a Dynkin diagram automorphism of $\mathfrak{g}$, and $\mathfrak{g}' = \mathfrak{g}^\sigma := \{X \in \mathfrak{g} | \sigma X = X\}$. We choose a σ-stable Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{h}^\sigma := \{X \in \mathfrak{h} | \sigma X = X\}$ is a Cartan subalgebra of $\mathfrak{g}^\sigma$. We shall use the same notation σ to denote the natural action on $\mathfrak{h}$, and also $\mathfrak{h}^*$. Let $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h})$ be the root system of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$, and Δ^+ be positive roots.

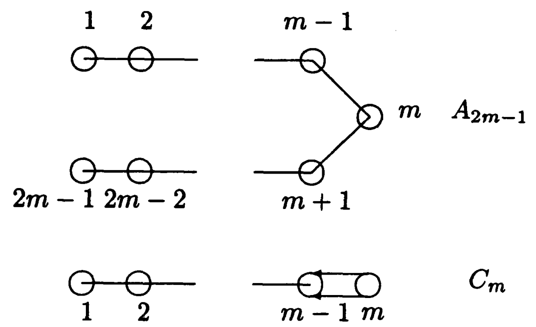
Case 1

$$(A_2, A_1) \quad \text{that is} \quad (\mathfrak{sl}(3,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C})).$$



Case 2

$$(A_{2m-1}, C_m) \quad (m \geq 2) \quad \text{that is} \quad (\mathfrak{sl}(2m,\mathbb{C}), \mathfrak{sp}(m,\mathbb{C})).$$



Case 3

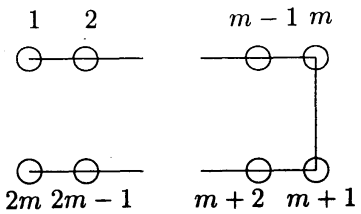
$$(A_{2m}, B_m) \quad (m \geq 2) \quad \text{that is} \quad (\mathfrak{sl}(2m+1,\mathbb{C}), \mathfrak{so}(2m+1,\mathbb{C})).$$

Case 4

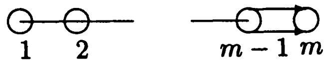
$$(D_m, B_{m-1}) \quad (m \geq 4) \quad \text{that is} \quad (\mathfrak{so}(2m,\mathbb{C}), \mathfrak{so}(2m-1,\mathbb{C})).$$

Case 5

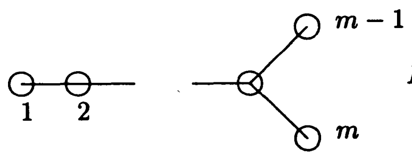
$$(E_6, F_4).$$



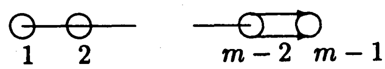
A_{2m}



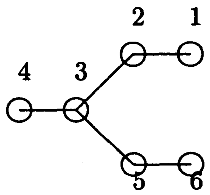
B_m



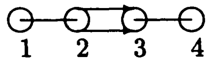
D_m



B_{m-1}



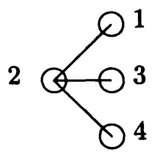
E_6



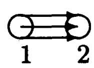
F_4

Case 6

(D_4, G_2).



D_4



G_2

Remark 6. We remark that there is a detailed study of $(\mathfrak{g}, \mathfrak{g}^\sigma)$ when $\mathfrak{g}$ is a generalized Kac-Moody Lie algebra by Fuchs-Schellekens-Schweigert [3] and Fuchs-Ray-Schweigert [4].

Remark 7. Only in Case 6, the order of σ is three. The pairs $(\mathfrak{g}, \mathfrak{g}^\sigma)$ in Cases 1-5 are called symmetric pairs.

3. Main Results

We denote by $X_n(\lambda)$ the irreducible finite dimensional representation of a complex simple Lie algebra of type X_n $(X = A, B, C, D, E, F, G)$ with a highest weight λ, and by $X_n(\lambda)|_{Y_{n'}}$ the restriction to a complex Lie algebra $\mathfrak{g}'$ of type $Y_{n'}$.

Let $\{\varpi_j\}_{j=1}^n$ be fundamental weights, with respect to a fixed simple system $\{\alpha_j\}_{j=1}^n$ of a complex Lie algebra of type X_n or $Y_{n'}$, which are labeled in the previous subsection.

Theorem 1. *For $k \in \mathbb{N}$,*

(2A) $$A_{2m-1}(k\varpi_1)|_{C_m} = A_{2m-1}(k\varpi_{2m-1})|_{C_m} = C_m(k\varpi_1) \quad (m \geq 2)$$

(4A) $$D_m(k\varpi_{m-1})|_{B_{m-1}} = D_m(k\varpi_m)|_{B_{m-1}} = B_{m-1}(k\varpi_{m-1}) \quad (m \geq 4)$$

Theorem 2. *For $k, l \in \mathbb{N}$,*

(1B) $$A_2(k\varpi_1)|_{A_1} = A_2(k\varpi_2)|_{A_1} = \bigoplus_{s=0}^{k} A_1(s\varpi_1)$$

(2B) $$A_{2m-1}(k\varpi_1 + l\varpi_2)|_{C_m} = A_{2m-1}(k\varpi_{2m-1} + l\varpi_{2m-2})|_{C_m} = \bigoplus_{s=0}^{l} C_m(k\varpi_1 + s\varpi_2) \quad (m \geq 3)$$

(3B) $$A_{2m}(k\varpi_1)|_{B_m} = A_{2m}(k\varpi_{2m})|_{B_m} = \bigoplus_{\substack{0 \leq s \leq k \\ s \equiv k \mod 2}} B_m(s\varpi_1) \quad (m \geq 2)$$

(4B) $$D_m(k\varpi_1)|_{B_{m-1}} = \bigoplus_{s=0}^{k} B_{m-1}(s\varpi_1) \quad (m \geq 4)$$

(5B) $$E_6(k\varpi_1)|_{F_4} = E_6(k\varpi_6)|_{F_4} = \bigoplus_{s=0}^{k} F_4(s\varpi_4).$$

$$(6B) \qquad D_4(k\varpi_1)|_{G_4} = D_4(k\varpi_3)|_{G_4} = D_4(k\varpi_4)|_{G_4} = \bigoplus_{s=0}^{k} G_2(s\varpi_2)$$

Remark 8. Some of these branching rules are new. One can prove some of them in several ways by using Borel-Weil theory, Gelfand-Tsetlin basis, formulas of minors, and so on (See, for example, [7], [14], [12], [11], [10]),

4. Sketch of Proof

We write down the sketch of proof of the theorems by using Weyl's character formula and denominator formula (See [2]).

Let $X_n(\lambda)$ be the representation which appears in left hand side of Theorems 1 and 2.

Let $\operatorname{char} X_n(\lambda)$ be the character of $X_n(\lambda)$.

We write $\rho_{X_n}, d_{X_n}, \Delta^+_{X_n}, W_{X_n}$ as half sum of positive roots, Weyl denominator, positive roots of complex simple Lie algebra of type X_n, Weyl group, respectively.

By Weyl's character formula,

$$\operatorname{char} X_n(\lambda) = d_{X_n}^{-1} \sum_{w \in W_{X_n}} \epsilon(w) e^{w(\lambda+\rho_{X_n})}$$

(We set $W_{X_n}(\lambda) := \{w \in W_{X_n} | w\lambda = \lambda\}$ and $W^\lambda_{X_n}$ minimal representatives of $W_{X_n}/W_{X_n}(\lambda)$.)

$$= d_{X_n}^{-1} \sum_{w_1 \in W^\lambda_{X_n}} \left(\sum_{w_2 \in W_{X_n}(\lambda)} \epsilon(w_1 w_2) e^{w_1 w_2 \lambda + w_1 w_2 \rho_{X_n}} \right)$$

$$= d_{X_n}^{-1} \sum_{w_1 \in W^\lambda_{X_n}} \left(\epsilon(w_1) e^{w_1 \lambda} \left(\sum_{w_2 \in W_{X_n}(\lambda)} \epsilon(w_2) e^{w_1 w_2 \rho_{X_n}} \right) \right)$$

By the denominator formula for $W_{X_n}(\lambda)$,

$$\sum_{w_2 \in W_{X_n}(\lambda)} \epsilon(w_2) e^{w_2 \rho_{X_n}} = e^{\rho_{X_n}} \prod_{\alpha \in \Delta^+_{X_n}(\lambda)} (1 - e^{-\alpha}).$$

Applying $w_1 \in W^{\lambda}_{X_n}$,

$$\sum_{w_2 \in W_{X_n}(\lambda)} \epsilon(w_2) e^{w_1 w_2 \rho_{X_n}} = e^{w_1 \rho_{X_n}} \prod_{\alpha \in \Delta^+_{X_n}(\lambda)} (1 - e^{-w_1 \alpha}).$$

Then,

(X)

$$\operatorname{char} X_n(\lambda) = d^{-1}_{X_n} \sum_{w_1 \in W^{\lambda}_{X_n}} \left(\epsilon(w_1) e^{w_1(\lambda)} \left(e^{w_1 \rho_{X_n}} \prod_{\alpha \in \Delta^+_{X_n}(\lambda)} (1 - e^{-w_1 \alpha}) \right) \right).$$

In the same way, we calculate $\operatorname{char} Y_{n'}(\lambda')$. $(\lambda' = \lambda|_{\mathfrak{h}^\sigma})$

(Y)

$$\operatorname{char} Y_{n'}(\lambda') = d^{-1}_{Y_{n'}} \sum_{w_1 \in W^{\lambda'}_{Y_{n'}}} \left(\epsilon(w_1) e^{w_1(\lambda')} \left(e^{w_1 \rho_{Y_{n'}}} \prod_{\alpha \in \Delta^+_{Y_{n'}}(\lambda')} (1 - e^{-w_1 \alpha}) \right) \right).$$

Lemma 3. *$W^{\lambda}_{X_n}$ and $W^{\lambda'}_{Y_{n'}}$ are "equal".*
In explicit, in the situation of Theorem 1, $W^{\lambda}_{X_n}$ and $W^{\lambda'}_{Y_{n'}}$ are equal. In the situation of Theorem 2, $W^{\lambda}_{X_n} \setminus W^{\lambda'}_{Y_{n'}}$ can be characterized by $w\varpi|_{\mathfrak{h}^\sigma} = 0$.

Remark 9. This lemma may be mysterious, because $W_{Y_{n'}}$ is much smaller than W_{X_n}.

Lemma 4. *The summands of* (X) *and* (Y) *are "equal".*
In explicit, in the situation of Theorem 1, the summands are equal. In the situation of Theorem 2, the difference of each summands is only one term.

We can prove the theorems by using the mysterious lemmas, in particular Lemma 3. We prove these lemmas by case-by-case calculation, then we do not understand why Lemma 3 is true.

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